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Stratified reduction of classical many-body systems with symmetry

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Abstract

The centre-of-mass system for N bodies admits a natural $SO(3)$ action, and thereby is stratified into strata according to the orbit types for the $SO(3)$ action. The principal stratum consists of non-singular configurations for which the isotropy subgroup is trivial, so that it is made into an $SO(3)$ principal fibre bundle. The strata of lower dimension consist of singular configurations at each of which the isotropy subgroup is not trivial. Practically, singular configurations are collinear ones and simultaneous multiple collision. Classical Lagrangian and Hamiltonian systems are defined on the tangent and the cotangent bundles over each stratum, respectively. The Euler–Lagrange and the Hamilton equations for the N-body system are derived on the variational principle, according to the stratification of the centre-of-mass system. The reduction procedure will be accordingly performed for the Lagrangian and the Hamiltonian systems with symmetry, respectively. By the rotational symmetry, the Euler–Lagrange and the Hamilton equations are reduced to those defined on reduced bundles from the tangent and the cotangent bundles, respectively.

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1. Introduction

This paper deals with the stratified reduction of classical systems for many bodies. A key idea to the stratified reduction is as follows: consider a manifold M and a compact Lie group G which acts on M, where M is supposed to be a configuration space for a classical system and the Lie group to describe symmetry of the classical system, respectively. According to the orbit types of the group action, the manifold M is stratified into strata. Classical mechanics will be set up on the tangent or the cotangent bundle over each stratum, and the reduction procedure will be performed for the classical system on each tangent or cotangent bundle on account of symmetry. The reduced system by symmetry and the reduced equations of motion are to be defined on vector bundles over respective quotient spaces of the strata.

This idea will be realized for a classical N-body system. Let M and G be the centre-of-mass system for N bodies and the rotation group $SO(3)$, respectively. Then M is stratified into strata according to the orbit types for the $SO(3)$ action. The principal stratum consists of non-singular configurations for which the isotropy subgroup is trivial, so that it is made into an $SO(3)$ principal fibre bundle. The strata of lower dimension consist of singular configurations at each of which the isotropy subgroup is not trivial. Practically, singular configurations are collinear ones and simultaneous multiple collision.

For non-singular configurations, one (T Iwai) of the authors performed the reduction procedure in the bundle picture [1–6] for classical and quantum mechanics of N-body systems. From a physical point of view, the gauge-theoretical reduction theory was set up for N-body systems in [7], while singular configurations have received little attention. However, mechanics for singular configurations will be able to be treated apart from non-singular configurations. In other words, mechanics will be set up on each stratum of the centre-of-mass system. Then the reduction procedure by symmetry will be carried out for classical systems on respective strata. The reduction procedure has already been performed for quantum N-body systems [8, 9]. This paper deals with the stratified reduction for classical N-body systems in both the Lagrangian and the Hamiltonian formalisms. The symplectic reduction for classical N-body systems was already studied in [1] for non-singular configurations.

A classical Lagrangian system is defined on the tangent bundle over the configuration space. The reduction procedure by symmetry has been investigated thoroughly in the Lagrangian formalism on the variational principle by Cendra, Marsden and Ratiu [10] (see also [11]). The Lagrangian reduction is related also to the study of non-holonomic mechanics [12, 13]. The cotangent bundle reduction [14] is in keeping with Hamiltonian mechanics. This article carries out the reduction procedure for N-body systems in both the Lagrangian and the Hamiltonian formalisms, but the reduction procedure is stratified according to the stratification of the centre-of-mass system. For geometric mechanics, the reference books [15–18] are of great help. The geometry in the calculus of variations is exposed in [19–21].

If the Lagrangian defined on the tangent bundle over each stratum is rotationally invariant, the Lagrangian system will reduce to a system defined on the factor space by $SO(3)$. The factor space is isomorphic with a vector bundle and will be referred to as the reduced bundle. The Euler–Lagrange equations on the tangent bundle over each stratum then reduce to those equations on the reduced bundle. In a similar manner, Hamilton's equations of motion defined on the cotangent bundle over each stratum are reduced to those on a reduced bundle by rotational symmetry. The Lagrangian and the Hamiltonian reduction procedures are comparable to each other. While the reduced equations in the Hamiltonian formalisms are described on the reduced bundle, this reduced bundle is not equal to the reduced phase space resulting from the original cotangent bundle in the process of symplectic reduction. The reduced bundle can be restricted to a subspace which is diffeomorphic with the reduced phase space.

This paper is organized as follows. Section 2 contains a brief review of geometric settings on the centre-of-mass system of N bodies. Local coordinate systems are introduced in the principal stratum (i.e., the space of non-singular configurations), and a connection form and a metric are defined and expressed in terms of the local coordinates. In section 3, the Euler–Lagrange equations are derived for the non-singular configurations on the variational principle. Further, for a rotationally-invariant Lagrangian, the reduced Euler–Lagrange equations are given. Section 4 deals with collinear configurations. The Euler–Lagrange equations are derived on the variational principle and then reduced with the rotational symmetry. In sections 5 and 6, the equations of motion are derived for the non-singular and for the collinear configurations, respectively, on the variational principle in the Hamiltonian formalism, and then reduced with the rotational symmetry. In section 7, the Euler–Lagrange equations for three-body systems are treated explicitly. It will be shown that by imposing some constraints on coordinates, velocities and components of the angular momentum, the equations of motion for non-singular configurations can reduce, in the limit, to those for collinear configurations. Section 8 contains remarks on the covariant derivatives which are used in describing reduced equations of motion, and on Hamel's approach to the Euler–Lagrange equation [22, 23].

2. Geometric settings

Following [9], we make a brief review of geometric settings on the centre-of-mass system. Getting rid of the translational degrees of freedom, we take the centre-of-mass system M for an N-body system, which is isomorphic with $\mathbf{R}^{3(N-1)}$;

$$M = \left\{ x = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \Big| \sum_{j=1}^{N} m_j \boldsymbol{x}_j = \mathbf{0} \right\} \cong \mathbf{R}^{3(N-1)}, \tag{2.1}$$

where $\boldsymbol{x}_j \in \mathbf{R}^3$ and $m_j, j = 1, \ldots, N$ are position vectors and masses of particles, respectively. We introduce Jacobi vectors, $\boldsymbol{r}_j, j = 1, \ldots, N-1$, which are defined to be

$$\boldsymbol{r}_j := \left(\frac{1}{\mu_j} + \frac{1}{m_{j+1}} \right)^{-\frac{1}{2}} \left(\boldsymbol{x}_{j+1} - \frac{1}{\mu_j} \sum_{i=1}^{j} m_i \boldsymbol{x}_i \right), \qquad \mu_j := \sum_{i=1}^{j} m_i. \tag{2.2}$$

Then, the isomorphism $M \cong \mathbf{R}^{3(N-1)}$ is realized as

$$M \cong \{x = (\boldsymbol{r}_1, \ldots, \boldsymbol{r}_{N-1}) | \boldsymbol{r}_j \in \mathbf{R}^3, j = 1, \ldots, N-1\}. \tag{2.3}$$

We call M a configuration space. When x is viewed as a $3 \times (N-1)$ matrix, according to its rank, the configuration space is decomposed into

$$M = M_0 \cup M_1 \cup M_2 \cup M_3, \tag{2.4}$$

$$M_k := \{x \in M | \operatorname{rank} x = k\}, \qquad k = 0, 1, 2, 3. \tag{2.5}$$

The rotation group $SO(3)$ acts on M in a natural manner;

$$x \longmapsto gx = (g\boldsymbol{r}_1, \ldots, g\boldsymbol{r}_{N-1}), \qquad g \in SO(3). \tag{2.6}$$

The $SO(3)$ action determines an equivalence relation on M, yielding a quotient space $M/SO(3)$. We denote by π the natural projection,

$$\pi : M \longrightarrow M/SO(3). \tag{2.7}$$

The space $M/SO(3)$ is called a shape space, which may fail to be a manifold.

We here denote the isotropy subgroup at $x \in M$ and the $SO(3)$-orbit through x by $G_x := \{g \in SO(3) | gx = x\}$ and by $\mathcal{O}_x := \{gx | g \in SO(3)\}$, respectively. Then one can verify that the isotropy subgroups G_x are classified as follows:

$$G_x = \begin{cases} \{e\} & \text{for} \quad x \in M_2 \cup M_3, \\ SO(2) & \text{for} \quad x \in M_1, \\ SO(3) & \text{for} \quad x \in M_0. \end{cases} \tag{2.8}$$

Accordingly, the orbits $\mathcal{O}_x$ are also classified into three:

$$\mathcal{O}_x \cong SO(3)/G_x \cong \begin{cases} SO(3) & \text{for} \quad x \in M_2 \cup M_3, \\ S^2 & \text{for} \quad x \in M_1, \\ \{0\} & \text{for} \quad x \in M_0. \end{cases} \tag{2.9}$$

Thus, M is stratified into a union of strata:

$$M = \dot{M} \cup M_1 \cup M_0. \tag{2.10}$$

A configuration $x \in M$ is called non-singular or singular, according as $x \in \dot{M}$ or $x \in \partial\dot{M} = M \backslash \dot{M}$. If rank $x = 0$, all particles collide at a point simultaneously, and if rank $x = 1$, the configuration of the particles is collinear.

Since each stratum is $SO(3)$ invariant, we are allowed to make M into a stratified fibre bundle with respective projections

$$\dot{M} \longrightarrow \dot{M}/SO(3), \qquad M_1 \longrightarrow M_1/SO(3), \qquad M_0 \longrightarrow M_0/SO(3) \tag{2.11}$$

whose fibres are $\mathcal{O}_x \cong SO(3)/G_x$, given by (2.9). In particular, $\dot{M}$ is made into a principal fibre bundle.

The inertia tensor, $A_x : \mathbf{R}^3 \to \mathbf{R}^3$, is defined through

$$A_x(\boldsymbol{v}) = \sum_{j=1}^{N-1} \boldsymbol{r}_j \times (\boldsymbol{v} \times \boldsymbol{r}_j), \qquad \boldsymbol{v} \in \mathbf{R}^3, \tag{2.12}$$

and the connection form ω is defined for $x \in \dot{M}$ to be

$$\omega = R\left(A_x^{-1}\left(\sum_{j=1}^{N-1} \boldsymbol{r}_j \times \mathrm{d}\boldsymbol{r}_j\right)\right), \tag{2.13}$$

where $R : \mathbf{R}^3 \to so(3)$ is the isomorphism defined by

$$R(\boldsymbol{a})\boldsymbol{x} = \boldsymbol{a} \times \boldsymbol{x}, \qquad \boldsymbol{a}, \boldsymbol{x} \in \mathbf{R}^3. \tag{2.14}$$

Note that A_x^{-1} exists only for $x \in \dot{M}$. The connection form ω determines a direct sum decomposition of the tangent space to $\dot{M}$ at $x \in \dot{M}$:

$$T_x(\dot{M}) = V_x \oplus H_x, \tag{2.15}$$

where $V_x := T_x(\mathcal{O}_x)$, the tangent space to $\mathcal{O}_x$, and $H_x := \ker \omega_x$ with $\omega_x : T_x(\dot{M}) \to so(3)$. Tangent vectors in V_x and in H_x are called rotational (or vertical) and vibrational (or horizontal), respectively. Further, V_x and H_x are orthogonal to each other with respect to the metric

$$\mathrm{d}s^2 = \sum_{j=1}^{N-1} \mathrm{d}\boldsymbol{r}_j \cdot \mathrm{d}\boldsymbol{r}_j. \tag{2.16}$$

We introduce local coordinates in $\dot{M}$ to express the connection form and the metric. Let σ be a local section defined on an open subset U of $\dot{Q}$; $\sigma : U \to \dot{M}$. Then any point $x \in \pi^{-1}(U)$ is expressed as

$$x = g\sigma(q) = (g\boldsymbol{\sigma}_1(q), \ldots, g\boldsymbol{\sigma}_{N-1}(q)), \qquad g \in SO(3), \quad q \in U. \tag{2.17}$$

Let $g \in SO(3)$ and $q \in U$ be assigned by the Euler angles (ϕ, θ, ψ) and by local shape coordinates $q^\alpha, \alpha = 1, \ldots, 3N - 6$, respectively. Then the connection form ω is put in the form

$$\omega_{g\sigma(q)} = \mathrm{d}g g^{-1} + g\omega_{\sigma(q)}g^{-1} = g(g^{-1}\,\mathrm{d}g + \omega_{\sigma(q)})g^{-1}, \tag{2.18}$$

where

$$\omega_{\sigma(q)} := R\left(A_{\sigma(q)}^{-1}\left(\sum_{j=1}^{N-1} \boldsymbol{\sigma}_j(q) \times \mathrm{d}\boldsymbol{\sigma}_j(q)\right)\right). \tag{2.19}$$

Let $\boldsymbol{e}_a, a = 1, 2, 3$, be the standard basis of $\mathbf{R}^3$. We introduce left-invariant 1-forms Ψ^a on $SO(3)$ by

$$g^{-1}\,\mathrm{d}g = \sum_{a=1}^{3} \Psi^a R(\boldsymbol{e}_a), \tag{2.20}$$

and define $\Lambda^a_\alpha(q)$ by expressing (2.19) as

$$\omega_{\sigma(q)} = \sum_{a=1}^{3}\sum_{\alpha=1}^{3N-6} \Lambda^a_\alpha(q)\,\mathrm{d}q^\alpha R(\boldsymbol{e}_a). \tag{2.21}$$

Then the connection form ω given by (2.18) takes the form

$$\omega_{g\sigma(q)} = \sum_{a=1}^{3} \Theta^a R(g\boldsymbol{e}_a), \qquad \Theta^a := \Psi^a + \sum_{\alpha=1}^{3N-6} \Lambda^a_\alpha(q)\,\mathrm{d}q^\alpha. \tag{2.22}$$

The horizontal lift, $\left(\frac{\partial}{\partial q^\alpha}\right)^*$, of a local vector field $\frac{\partial}{\partial q^\alpha}$ on U is shown to be given by

$$\left(\frac{\partial}{\partial q^\alpha}\right)^* = \frac{\partial}{\partial q^\alpha} - \sum_{a=1}^{3} \Lambda^a_\alpha(q) K_a, \qquad \alpha = 1, 2, \ldots, 3N-6, \tag{2.23}$$

where K_a denote the left-invariant vector fields on $SO(3)$, which are dual to Ψ^a. The $\mathrm{d}q^\alpha$, Θ^a and the $\left(\frac{\partial}{\partial q^\alpha}\right)^*$, K_a form local bases of 1-forms and of vector fields on $\pi^{-1}(U) \cong U \times SO(3)$, respectively. They are dual to each other.

According to the orthogonal decomposition (2.15), we can express metric (2.16) in terms of $\mathrm{d}q^\alpha$, Θ^a as

$$\mathrm{d}s^2 = \sum_{\alpha,\beta} a_{\alpha\beta}\,\mathrm{d}q^\alpha\,\mathrm{d}q^\beta + \sum_{a,b} A_{ab}\Theta^a\Theta^b, \tag{2.24}$$

where we have introduced the notation $a_{\alpha\beta}$ and A_{ab} by

$$a_{\alpha\beta} := \mathrm{d}s^2\left(\left(\frac{\partial}{\partial q^\alpha}\right)^*, \left(\frac{\partial}{\partial q^\beta}\right)^*\right), \tag{2.25}$$

$$A_{ab} := \boldsymbol{e}_a \cdot A_{\sigma(q)}(\boldsymbol{e}_b) = \mathrm{d}s^2(K_a, K_b). \tag{2.26}$$

We note here that $(a_{\alpha\beta})$ defines a Riemannian metric on $\dot{M}/SO(3)$.

In conclusion of this section, we give the transformation law for local expressions of the connection form and of the inertia tensor. Let $\sigma' : U' \to \dot{M}$ be another local section with $U' \cap U \neq \emptyset$. Then there exists an $SO(3)$-valued function $h(q) \in SO(3)$ such that $\sigma'(q) = h(q)\sigma(q)$, $q \in U' \cap U$. From (2.18), it then follows that

$$\omega_{\sigma'(q)} = \mathrm{d}h\,h^{-1} + h\omega_{\sigma(q)}h^{-1}, \tag{2.27}$$

which provides the transformation law

$$\Lambda'_\alpha(q) = \frac{\partial h}{\partial q^\alpha}h^{-1} + h\Lambda_\alpha(q)h^{-1}, \tag{2.28}$$

where

$$\omega_{\sigma'(q)} = \sum_\alpha \Lambda'_\alpha(q)\,\mathrm{d}q^\alpha, \qquad \omega_{\sigma(q)} = \sum_\alpha \Lambda_\alpha(q)\,\mathrm{d}q^\alpha. \tag{2.29}$$

Moreover, the transformation law for the inertia tensor $A = (A_{ab})$ is given by

$$A' = hAh^{-1}, \tag{2.30}$$

where

$$A' = (A'_{ab}), \qquad A'_{ab} := \boldsymbol{e}_a \cdot A_{\sigma'(q)}(\boldsymbol{e}_b). \tag{2.31}$$

3. Lagrangian mechanics for non-singular configurations

In this section, we derive Euler–Lagrange equations for the non-singular configurations on the variational principle. Further, for a rotationally-invariant Lagrangian, we obtain reduced Euler–Lagrange equations by the use of $SO(3)$ symmetry.

Let $(q, g, \dot{q}, \dot{g})$ be local coordinates on $T(\pi^{-1}(U))$, where $(q, g) \in \pi^{-1}(U)$ and $(\dot{q}, \dot{g}) \in T_{\sigma(q)}(\pi^{-1}(U))$ with $q = (q^\alpha), \dot{q} = (\dot{q}^\alpha)$. In view of the connection form Θ^a given by (2.22), we introduce an $so(3)$-valued variable by

$$\Pi = \xi + \sum_{\alpha=1}^{3N-6} \Lambda_\alpha(q)\dot{q}^\alpha, \tag{3.1}$$

where

$$\xi = g^{-1}\dot{g}, \qquad \Lambda_\alpha(q) = \sum_{a=1}^{3} \Lambda_\alpha^a(q) R(e_a). \tag{3.2}$$

We further denote by $\boldsymbol{\pi}$ the vector associated with Π,

$$R(\boldsymbol{\pi}) = \Pi. \tag{3.3}$$

We take $(q, g, \dot{q}, \Pi)$ as local coordinates in $T(\pi^{-1}(U))$. Assume that we are given a Lagrangian $L(q, g, \dot{q}, \Pi)$. We wish to obtain the Euler–Lagrange equations for L on the basis of the variational principle, which will be determined by

$$\delta \int_{t_1}^{t_2} L(q, g, \dot{q}, \Pi)\, \mathrm{d}t = 0 \tag{3.4}$$

with the boundary conditions

$$\delta q(t_i) = 0, \qquad \delta g(t_i) = 0, \qquad i = 1, 2. \tag{3.5}$$

The variation of L is expressed as

$$\delta L = \sum_\alpha \frac{\partial L}{\partial q^\alpha}\delta q^\alpha + \sum_\alpha \frac{\partial L}{\partial \dot{q}^\alpha}\delta \dot{q}^\alpha + \left\langle \frac{\partial L}{\partial g}, \delta g \right\rangle + \left\langle \frac{\partial L}{\partial \Pi}, \delta \Pi \right\rangle, \tag{3.6}$$

where we have denoted the inner product of 3×3 matrices A and B by

$$\langle A, B \rangle := \mathrm{tr}(A^T B). \tag{3.7}$$

Note here that $\frac{\partial L}{\partial \Pi}$ and $\frac{\partial L}{\partial \boldsymbol{\pi}}$ are related by

$$R\left(\frac{\partial L}{\partial \boldsymbol{\pi}}\right) = 2\frac{\partial L}{\partial \Pi}, \tag{3.8}$$

which implies that

$$\frac{\partial L}{\partial \boldsymbol{\pi}} \cdot \delta\boldsymbol{\pi} = \left\langle \frac{\partial L}{\partial \Pi}, \delta \Pi \right\rangle. \tag{3.9}$$

Since the variation $\delta\Pi$ is put in the form

$$\delta\Pi = [\xi, g^{-1}\delta g] + \frac{\mathrm{d}}{\mathrm{d}t}(g^{-1}\delta g) + \sum_{\alpha,\beta}\left(\frac{\partial \Lambda_\beta}{\partial q^\alpha} - \frac{\partial \Lambda_\alpha}{\partial q^\beta}\right)\dot{q}^\beta \delta q^\alpha + \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_\alpha \Lambda_\alpha \delta q^\alpha\right), \tag{3.10}$$

and since the variation of L with respect to g is brought into the form

$$\left\langle \frac{\partial L}{\partial g}, \delta g \right\rangle = \frac{1}{2}\left\langle g^{-1}\frac{\partial L}{\partial g} - \left(\frac{\partial L}{\partial g}\right)^T g, g^{-1}\delta g \right\rangle, \tag{3.11}$$

it turns out after calculation that the Euler–Lagrange equations are given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^\alpha}\right) - \frac{\partial L}{\partial q^\alpha} = \left\langle \frac{\partial L}{\partial \Pi}, \sum_\beta K_{\alpha\beta}\dot{q}^\beta \right\rangle - \left\langle \frac{\partial L}{\partial \Pi}, [\Pi, \Lambda_\alpha] \right\rangle$$
$$- \frac{1}{2}\left\langle g^{-1}\frac{\partial L}{\partial g} - \left(\frac{\partial L}{\partial g}\right)^T g, \Lambda_\alpha \right\rangle, \tag{3.12a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \Pi} = \left[\frac{\partial L}{\partial \Pi}, \Pi\right] - \sum_\beta \left[\frac{\partial L}{\partial \Pi}, \Lambda_\beta\right]\dot{q}^\beta + \frac{1}{2}\left(g^{-1}\frac{\partial L}{\partial g} - \left(\frac{\partial L}{\partial g}\right)^T g\right), \tag{3.12b}$$

where $K_{\alpha\beta}$ are the curvature tensors defined by

$$K_{\alpha\beta} := \frac{\partial \Lambda_\beta}{\partial q^\alpha} - \frac{\partial \Lambda_\alpha}{\partial q^\beta} - [\Lambda_\alpha, \Lambda_\beta]. \tag{3.13}$$

We now point out that the Euler–Lagrange equations are described independently of the choice of local sections. Let $\sigma' : U' \to \dot{M}$ be another local section such that $\sigma'(q) = h(q)\sigma(q), h(q) \in SO(3)$, for $q \in U' \cap U \neq \emptyset$. As referred to in the previous section, the Λ_α and $A_{\sigma(q)}$ transform according to (2.28) and (2.30), respectively. From the transformation law for Λ_α, one can verify that the transformation laws of the curvature tensor and of the variable Π are given by

$$K'_{\alpha\beta} = h(q)K_{\alpha\beta}h^{-1}(q), \tag{3.14}$$

$$\Pi' = h(q)\Pi h^{-1}(q), \tag{3.15}$$

respectively. Since $L(q, g, \dot{q}, \Pi) = L'(q, g', \dot{q}, \Pi')$ on $T(U \cap U')$, one verifies that the Euler–Lagrange equations (3.12) are described independently of the choice of local sections.

Proposition 3.1. *The Euler–Lagrange equations for the non-singular configurations are given by (3.12), which are independent of the choice of local sections $U \to \dot{M}$.*

Assume now that L is invariant under the left $SO(3)$ action, i.e., L is rotationally invariant,

$$L(q, \dot{q}, hg, \Pi) = L(q, \dot{q}, g, \Pi) \qquad \text{for all} \quad h \in SO(3). \tag{3.16}$$

Note here that Π is left-invariant. Then this equation with $h = \mathrm{e}^{t\eta}$, $\eta \in so(3)$, is differentiated with respect to t at $t = 0$ to provide

$$\frac{\mathrm{d}}{\mathrm{d}t}L(q, \dot{q}, \mathrm{e}^{t\eta}g, \Pi)\bigg|_{t=0} = \left\langle \frac{\partial L}{\partial g}, \eta g \right\rangle = \frac{1}{2}\left\langle \frac{\partial L}{\partial g}g^{-1} - g\left(\frac{\partial L}{\partial g}\right)^T, \eta \right\rangle = 0. \tag{3.17}$$

Since $\eta \in so(3)$ is arbitrary, one obtains $\frac{\partial L}{\partial g}g^{-1} - g\left(\frac{\partial L}{\partial g}\right)^T = 0$, so that

$$g^{-1}\frac{\partial L}{\partial g} - \left(\frac{\partial L}{\partial g}\right)^T g = 0. \tag{3.18}$$

From (3.16), L will reduce to a function $L^*(q, \dot{q}, \Pi)$ on the reduced bundle

$$T\dot{M}/SO(3) \cong T(\dot{M}/SO(3)) \oplus \widetilde{\mathcal{G}}, \tag{3.19}$$

where the right-hand side is a Whitney sum bundle, and $\widetilde{\mathcal{G}}$ denotes the vector bundle associated with the adjoint action of $SO(3)$ on $\mathcal{G}$, $\widetilde{\mathcal{G}} := \dot{M} \times_{SO(3)} \mathcal{G}$ with $\mathcal{G} = so(3)$ (see [10, 11] for the reduced bundle). From (3.12) and (3.18), the reduced Euler–Lagrange equations take the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L^*}{\partial \dot{q}^\alpha}\right) - \frac{\partial L^*}{\partial q^\alpha} = \left\langle \frac{\partial L^*}{\partial \Pi}, \sum_\beta K_{\alpha\beta}\dot{q}^\beta \right\rangle - \left\langle \frac{\partial L^*}{\partial \Pi}, [\Pi, \Lambda_\alpha] \right\rangle, \tag{3.20a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L^*}{\partial \Pi} = \left[\frac{\partial L^*}{\partial \Pi}, \Pi\right] - \sum_\beta \left[\frac{\partial L^*}{\partial \Pi}, \Lambda_\beta\right]\dot{q}^\beta. \tag{3.20b}$$

From (3.20*b*), which is also put in the form $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L^*}{\partial \Pi} = \left[\frac{\partial L^*}{\partial \Pi}, \xi\right]$, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(g\frac{\partial L^*}{\partial \Pi}g^{-1}\right) = 0. \tag{3.21}$$

Proposition 3.2. *If the Lagrangian is rotationally invariant, the Euler–Lagrange equations for the non-singular configurations reduce to (3.20), which are defined on the reduced bundle (3.19). The reduced Euler–Lagrange equations are independent of the choice of local sections* $U \to \dot{M}$. *Further, the quantity* $g\frac{\partial L^*}{\partial \Pi}g^{-1}$ *is conserved.*

For non-singular configurations, a rotationally-invariant Lagrangian is given, from the kinetic metric (2.24), by

$$L^* = \frac{1}{2}\sum_{\alpha,\beta} a_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta + \frac{1}{2}\sum_{a,b} A_{ab}\pi^a\pi^b - V(q), \tag{3.22}$$

where $a_{\alpha\beta}$ and A_{ab} are, respectively, the Riemannian metric and the inertia tensor given by (2.25) and (2.26), and π^a are the components of $\Pi, \Pi = \sum_a \pi^a R(\boldsymbol{e}_a)$, and where $V(q)$ denotes a potential function depending on q only. We now rewrite the right-hand side of the above equation in terms of Π with $R(\boldsymbol{\pi}) = \Pi$. On introducing the inertia tensor on $so(3)$ by

$$\widetilde{A}_{\sigma(q)} := RA_{\sigma(q)}R^{-1}, \tag{3.23}$$

the Lagrangian is put in the form

$$L^* = \frac{1}{2}\sum_{\alpha,\beta} a_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta + \frac{1}{4}\langle \Pi, \widetilde{A}_{\sigma(q)}\Pi\rangle - V(q). \tag{3.24}$$

Then, the reduced Euler–Lagrange equations (3.20) turn out to be expressed, in vector notation, as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\dot{q}^\alpha &+ \sum_{\beta,\gamma}\Gamma^\alpha_{\beta\gamma}\dot{q}^\beta\dot{q}^\gamma + \sum_\beta a^{\alpha\beta}\frac{\partial V}{\partial q^\beta} \\ &= \sum_{\beta,\gamma} a^{\alpha\beta}\left(A_{\sigma(q)}\boldsymbol{\pi}\cdot\boldsymbol{\kappa}_{\beta\gamma}\dot{q}^\gamma + \frac{1}{2}\boldsymbol{\pi}\cdot\left(\frac{\partial A_{\sigma(q)}}{\partial q^\beta} - [\Lambda_\beta, A_{\sigma(q)}]\right)\boldsymbol{\pi}\right),\end{aligned} \tag{3.25a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(A_{\sigma(q)}\boldsymbol{\pi}) = A_{\sigma(q)}\boldsymbol{\pi}\times\boldsymbol{\pi} - \sum_\alpha A_{\sigma(q)}\boldsymbol{\pi}\times\boldsymbol{\lambda}_\alpha\dot{q}^\alpha, \tag{3.25b}$$

where $\boldsymbol{\kappa}_{\alpha\beta}$ and $\boldsymbol{\lambda}_\alpha$ are vectors determined through

$$K_{\alpha\beta} = R(\boldsymbol{\kappa}_{\alpha\beta}), \qquad \Lambda_\alpha = R(\boldsymbol{\lambda}_\alpha), \tag{3.26}$$

and where $\Gamma^{\alpha}_{\beta\gamma}$ denote the Christoffel symbols formed from $a_{\alpha\beta}$,

$$\Gamma^{\alpha}_{\beta\gamma} := \frac{1}{2}\sum_{\delta} a^{\alpha\delta}\left(\frac{\partial a_{\delta\gamma}}{\partial q^{\beta}} + \frac{\partial a_{\delta\beta}}{\partial q^{\gamma}} - \frac{\partial a_{\beta\gamma}}{\partial q^{\delta}}\right), \qquad (a^{\alpha\beta}) = (a_{\alpha\beta})^{-1}. \tag{3.27}$$

Proposition 3.3. *For non-singular configurations, the reduced Euler–Lagrange equations for Lagrangian (3.24) are expressed, in vector notation, as (3.25a) and (3.25b). Note that the right-hand side of equation (3.25a) describes a generalized Lorentz force and a centrifugal force. Equation (3.25b) is a generalization of the Euler equation for a rigid body to one for deformable configurations.*

We note here that the conserved quantity $g\frac{\partial L^*}{\partial \Pi}g^{-1}$ is half the total angular momentum. In fact, one has

$$g\frac{\partial L^*}{\partial \Pi}g^{-1} = \frac{1}{2}gR(A_{\sigma(q)}\boldsymbol{\pi})g^{-1} = \frac{1}{2}R(gA_{\sigma(q)}\boldsymbol{\pi}) = \frac{1}{2}R(\boldsymbol{L}), \tag{3.28}$$

where

$$\boldsymbol{L} = \sum_j \boldsymbol{r}_j \times \dot{\boldsymbol{r}}_j = gA_{\sigma(q)}\boldsymbol{\pi}, \tag{3.29}$$

as is easily shown.

We further make remarks on covariant derivatives appearing in (3.25). From the transformation laws (2.28) and (2.30), it follows that

$$\frac{\partial A'}{\partial q^{\alpha}} - [\Lambda'_{\alpha}, A'] = h(q)\left(\frac{\partial A}{\partial q^{\alpha}} - [\Lambda_{\alpha}, A]\right)h^{-1}(q), \tag{3.30}$$

which implies that $\frac{\partial A}{\partial q^{\alpha}} - [\Lambda_{\alpha}, A]$ is the covariant derivative of A with respect to $\frac{\partial}{\partial q^{\alpha}}$. The covariant derivation along a curve $q(t)$ is defined by

$$\frac{\mathrm{D}}{\mathrm{d}t} := \frac{\mathrm{d}}{\mathrm{d}t} - \sum_{\alpha}\Lambda_{\alpha}\dot{q}^{\alpha} = \sum_{\alpha}\dot{q}^{\alpha}\frac{\mathrm{D}}{\partial q^{\alpha}}, \tag{3.31}$$

so that equation (3.25*b*) can be expressed as

$$\frac{\mathrm{D}}{\mathrm{d}t}(A_{\sigma(q)}\boldsymbol{\pi}) = A_{\sigma(q)}\boldsymbol{\pi} \times \boldsymbol{\pi}. \tag{3.32}$$

In section 8, we will make remarks on covariant derivation in more detail.

We conclude this section with a comment on the expression of our Lagrangian. If we take $(q, g, \dot{q}, \xi)$ as local coordinates of $T\dot{M}$, then the Lagrangian is put in the form

$$L^* = \frac{1}{2}\sum_{\alpha,\beta} a_{\alpha\beta}\dot{q}^{\alpha}\dot{q}^{\beta} + \frac{1}{2}\left(\boldsymbol{\Omega} + \sum_{\alpha}\boldsymbol{\lambda}_{\alpha}\dot{q}^{\alpha}\right)\cdot A_{\sigma(q)}\left(\boldsymbol{\Omega} + \sum_{\alpha}\boldsymbol{\lambda}_{\alpha}\dot{q}^{\alpha}\right) - V(q), \tag{3.33}$$

where $R(\boldsymbol{\Omega}) = \xi$ and $\boldsymbol{\pi} = \boldsymbol{\Omega} + \sum_{\alpha}\boldsymbol{\lambda}_{\alpha}\dot{q}^{\alpha}$. This description of the Lagrangian is usually adopted in the physics literature [7].

4. Lagrangian mechanics for collinear configurations

In this section, we work with the space of collinear configurations M_1. As referred to in section 2, the orbit of $SO(3)$ through $x \in M_1$ is identified with S^2. Though M_1 is not a principal fibre bundle, we can decompose the tangent space to M_1 at $x \in M_1$ into a direct sum of vertical and horizontal subspaces; the vertical subspace $V_x^{(1)}$ is defined to be the tangent space to the orbit $\mathcal{O}_x$, and the horizontal subspace $H_x^{(1)}$ to be the orthogonal complement of $V_x^{(1)}$;

$$T_x(M_1) = V_x^{(1)} \oplus H_x^{(1)}, \qquad V_x^{(1)} := T_x(\mathcal{O}_x), \qquad H_x^{(1)} := \left(V_x^{(1)}\right)^{\perp}, \tag{4.1}$$

where the metric on M_1 with respect to which the orthogonality is referred is induced from that on the centre-of-mass system M.

Let $\widetilde{\sigma}(\zeta)$ be a local section in M_1; $\widetilde{\sigma} : \widetilde{U} \subset M_1/S^2 \to M_1$, which is given by

$$\widetilde{\sigma}(\zeta) = (\zeta^1 \boldsymbol{e}_3, \ldots, \zeta^{N-1}\boldsymbol{e}_3), \qquad \zeta = (\zeta^\alpha) \in \widetilde{U}, \tag{4.2}$$

where ζ^α are local shape coordinates in $\widetilde{U} \subset M_1/S^2$. Then a generic point $x \in M_1$ is expressed as

$$x = g\widetilde{\sigma}(\zeta) = (\zeta^1 g\boldsymbol{e}_3, \ldots, \zeta^{N-1} g\boldsymbol{e}_3), \qquad g \in SO(3). \tag{4.3}$$

We express g as $g = \mathrm{e}^{\phi R(e_3)}\mathrm{e}^{\theta R(e_2)}\mathrm{e}^{\psi R(e_3)}$. Then x is assigned by local coordinates $(\theta, \phi, \zeta^1, \ldots, \zeta^{N-1})$, ψ being eliminated. Hence, we may view g as $g = \mathrm{e}^{\phi R(e_3)}\mathrm{e}^{\theta R(e_2)}$ in this section. Now the induced metric $\widetilde{\mathrm{d}s}^2$ on M_1 is expressed as

$$\widetilde{\mathrm{d}s}^2 = \sum_{\alpha=1}^{N-1} (\zeta^\alpha)^2(\sin^2\theta\, \mathrm{d}\phi^2 + \mathrm{d}\theta^2) + \sum_{\alpha=1}^{N-1} (\mathrm{d}\zeta^\alpha)^2. \tag{4.4}$$

See [9] for details.

Let $(\zeta, \boldsymbol{u}, \dot{\zeta}, \dot{\boldsymbol{u}})$ be local coordinates in the tangent bundle TM_1, where $\zeta = (\zeta^\alpha)$, $\dot{\zeta} = (\dot{\zeta}^\alpha)$ and where

$$\boldsymbol{u} = g\boldsymbol{e}_3, \qquad g = \mathrm{e}^{\phi R(e_3)}\mathrm{e}^{\theta R(e_2)}. \tag{4.5}$$

By introducing an $so(3)$-valued varable $\widetilde{\xi}$ and a vector-valued variable $\widetilde{\boldsymbol{\Omega}}$ through

$$\widetilde{\xi} = g^{-1}\dot{g} = R(\widetilde{\boldsymbol{\Omega}}), \tag{4.6}$$

the $\dot{\boldsymbol{u}}$ is expressed also as

$$\dot{\boldsymbol{u}} = g\widetilde{\xi}\boldsymbol{e}_3 = gR(\widetilde{\boldsymbol{\Omega}})\boldsymbol{e}_3 = g(\widetilde{\boldsymbol{\Omega}} \times \boldsymbol{e}_3). \tag{4.7}$$

In view of this, we introduce an $so(3)$-valued variable $\widetilde{\Pi}$ by

$$\widetilde{\Pi} := R(\widetilde{\boldsymbol{\Omega}} \times \boldsymbol{e}_3) = [g^{-1}\dot{g}, R(\boldsymbol{e}_3)]. \tag{4.8}$$

We take $(\zeta, \boldsymbol{u}, \dot{\zeta}, \widetilde{\Pi})$ as local coordinates in TM_1. We wish to derive the Euler–Lagrange equation for a Lagrangian $L = L(\zeta, \boldsymbol{u}, \dot{\zeta}, \widetilde{\Pi})$ on the variational principle. The variation of L is put in the form

$$\delta L = \sum_\alpha \frac{\partial L}{\partial \zeta^\alpha}\delta\zeta^\alpha + \sum_\alpha \frac{\partial L}{\partial \dot{\zeta}^\alpha}\delta\dot{\zeta}^\alpha + \frac{\partial L}{\partial \boldsymbol{u}} \cdot \delta\boldsymbol{u} + \left\langle \frac{\partial L}{\partial \widetilde{\Pi}}, \delta\widetilde{\Pi} \right\rangle. \tag{4.9}$$

Since $|\boldsymbol{u}| = 1$, the infinitesimal variation $\delta\boldsymbol{u}$ should be restricted. We can rewrite the variation of L with respect to $\boldsymbol{u}$ as

$$\frac{\partial L}{\partial \boldsymbol{u}} \cdot \delta\boldsymbol{u} = \frac{1}{2}\left\langle \widetilde{P}\left(R\left(g^{-1}\frac{\partial L}{\partial \boldsymbol{u}}\right)\right), [g^{-1}\delta g, R(\boldsymbol{e}_3)] \right\rangle, \tag{4.10}$$

where $\widetilde{P}$ is the projection operator defined on $so(3)$ through

$$\widetilde{P}R = RP, \qquad P := I - \boldsymbol{e}_3\boldsymbol{e}_3^T. \tag{4.11}$$

Alternatively, $\widetilde{P}$ is given by

$$\widetilde{P}(\eta) = [R(\boldsymbol{e}_3), [\eta, R(\boldsymbol{e}_3)]], \qquad \text{for} \quad \eta \in so(3). \tag{4.12}$$

Since

$$2\boldsymbol{u}\cdot\delta\boldsymbol{u} = 2\boldsymbol{e}_3\cdot g^{-1}\delta g\boldsymbol{e}_3 = \langle R(\boldsymbol{e}_3), [g^{-1}\delta g, R(\boldsymbol{e}_3)]\rangle = 0, \tag{4.13}$$

the constraint $|\boldsymbol{u}| = 1$ has been taken into account in (4.10). We note further that

$$[g^{-1}\delta g, R(\boldsymbol{e}_3)] = \sin\theta\delta\phi R(\boldsymbol{e}_2) + \delta\theta R(\boldsymbol{e}_1), \tag{4.14}$$

which shows that the variation $[g^{-1}\delta g, R(\boldsymbol{e}_3)]$ may take arbitrary values in $\widetilde{P}so(3)$. The variation $\delta\widetilde{\Pi}$ is put in the form

$$\delta\widetilde{\Pi} = [[\widetilde{\xi}, g^{-1}\delta g], R(\boldsymbol{e}_3)] + \left[\frac{\mathrm{d}}{\mathrm{d}t}(g^{-1}\delta g), R(\boldsymbol{e}_3)\right]. \tag{4.15}$$

This means that $\widetilde{P}(\delta\widetilde{\Pi}) = \delta\widetilde{\Pi}$, i.e., the $R(\boldsymbol{e}_3)$-component of $\delta\widetilde{\Pi}$ vanishes. From (4.10) and (4.15), the Euler–Lagrange equations prove to be given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\zeta}^\alpha}\right) - \frac{\partial L}{\partial\zeta^\alpha} = 0, \tag{4.16a}$$

$$\widetilde{P}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\widetilde{\Pi}} - \left[\widetilde{P}\left(\frac{\partial L}{\partial\widetilde{\Pi}}\right), \widetilde{\xi}\right] - \frac{1}{2}R\left(g^{-1}\frac{\partial L}{\partial\boldsymbol{u}}\right)\right) = 0, \tag{4.16b}$$

$$\left[\widetilde{P}\left(\frac{\partial L}{\partial\widetilde{\Pi}}\right), \widetilde{\Pi}\right] = 0, \tag{4.16c}$$

where (4.16c) is a constraint to be required. There are two ways to take local sections $\widetilde{\sigma} : \widetilde{U} \to M_1$; one is $\widetilde{\sigma}(\zeta) = (\zeta^1\boldsymbol{e}_3, \ldots, \zeta^{N-1}\boldsymbol{e}_3)$, and the other $\widetilde{\sigma}'(\zeta') = (-\zeta'^1\boldsymbol{e}_3, \ldots, -\zeta'^{N-1}\boldsymbol{e}_3)$. Hence, the transformation law is given by $\zeta'^\alpha = -\zeta^\alpha$. Since metric (4.4) is invariant under the inversion $\zeta^\alpha \mapsto -\zeta^\alpha$, $\dot{\zeta}^\alpha \mapsto -\dot{\zeta}^\alpha$, so is equation (4.16).

Proposition 4.1. *The Euler–Lagrange equations for the collinear configurations are given by (4.16), which are independent of the choice of local sections* $\widetilde{U} \to M_1$.

We now assume that the Lagrangian is rotationally invariant,

$$L(\zeta, \mathrm{e}^{tR(\boldsymbol{a})}\boldsymbol{u}, \dot{\zeta}, \widetilde{\Pi}) = L(\zeta, \boldsymbol{u}, \dot{\zeta}, \widetilde{\Pi}) \qquad \text{for all} \quad R(\boldsymbol{a}) \in so(3). \tag{4.17}$$

Note here that $\widetilde{\Pi}$ is left-invariant. From (4.17), it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}L(\zeta, \mathrm{e}^{tR(\boldsymbol{a})}\boldsymbol{u}, \dot{\zeta}, \widetilde{\Pi})|_{t=0} = R(\boldsymbol{a})\boldsymbol{u}\cdot\frac{\partial L}{\partial\boldsymbol{u}} = \boldsymbol{a}\cdot\left(\boldsymbol{u}\times\frac{\partial L}{\partial\boldsymbol{u}}\right) = 0, \tag{4.18}$$

so that

$$\boldsymbol{u}\times\frac{\partial L}{\partial\boldsymbol{u}} = g\left(\boldsymbol{e}_3\times g^{-1}\frac{\partial L}{\partial\boldsymbol{u}}\right) = 0. \tag{4.19}$$

This implies that

$$\widetilde{P}\left(R\left(g^{-1}\frac{\partial L}{\partial\boldsymbol{u}}\right)\right) = 0. \tag{4.20}$$

From (4.17), L will reduce to a function $L^*(\zeta, \dot{\zeta}, \widetilde{\Pi})$ on the reduced bundle

$$TM_1/SO(3) \cong T(M_1/SO(3)) \oplus \widetilde{\mathcal{G}}_1, \tag{4.21}$$

where $\widetilde{\mathcal{G}}_1$ is a vector bundle over $M_1/SO(3)$ with the standard fibre $\widetilde{P}so(3)$. Owing to (4.20), the Euler–Lagrange equations (4.16) reduce to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L^*}{\partial\dot{\zeta}^\alpha}\right) - \frac{\partial L^*}{\partial\zeta^\alpha} = 0, \tag{4.22a}$$

$$\widetilde{P}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L^*}{\partial \widetilde{\Pi}} - \left[\widetilde{P}\left(\frac{\partial L^*}{\partial \widetilde{\Pi}}\right), \widetilde{\xi}\right]\right) = 0, \tag{4.22b}$$

$$\left[\widetilde{P}\left(\frac{\partial L^*}{\partial \widetilde{\Pi}}\right), \widetilde{\Pi}\right] = 0. \tag{4.22c}$$

We now consider a conservation law in an analogous manner to the case of non-singular configurations. Equations (4.22*b*), (4.22*c*) give rise to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(g\left[\widetilde{P}\left(\frac{\partial L^*}{\partial \widetilde{\Pi}}\right), R(\boldsymbol{e}_3)\right]g^{-1}\right) = 0, \tag{4.23}$$

which implies that the angular momentum is conserved, as will be shown soon.

Proposition 4.2. *If the Lagrangian is rotationally invariant, the Euler–Lagrange equations for the collinear configurations reduce to (4.22), which are defined on the reduced bundle (4.21). Further, the quantity* $g\big[\widetilde{P}\big(\frac{\partial L^*}{\partial \widetilde{\Pi}}\big), R(\boldsymbol{e}_3)\big]g^{-1}$ *is conserved.*

From (4.4), a Lagrangian for collinear configurations is shown to be given by

$$L^* = \frac{1}{2}\sum_\alpha (\dot{\zeta}^\alpha)^2 + \frac{1}{2}\rho(\zeta)|\widetilde{\boldsymbol{\Omega}} \times \boldsymbol{e}_3|^2 - V_1(\zeta), \qquad \rho(\zeta) = \sum_\alpha (\zeta^\alpha)^2, \tag{4.24}$$

where V_1 is a potential function. This Lagrangian is rotationally invariant. Rewriting the second term of the right-hand side of the above equation in terms of $\widetilde{\Pi}$, we obtain the Lagrangian in the form

$$L^* = \frac{1}{2}\sum_\alpha (\dot{\zeta}^\alpha)^2 + \frac{1}{4}\rho(\zeta)\langle\widetilde{\Pi}, \widetilde{\Pi}\rangle - V_1(\zeta). \tag{4.25}$$

Since $\widetilde{P}\big(\frac{\partial L^*}{\partial \widetilde{\Pi}}\big) = \frac{1}{2}\widetilde{P}(\rho(\zeta)\widetilde{\Pi}) = \frac{1}{2}\rho(\zeta)\widetilde{\Pi}$, one has

$$\left[\widetilde{P}\left(\frac{\partial L^*}{\partial \widetilde{\Pi}}\right), \widetilde{\Pi}\right] = 0. \tag{4.26}$$

The Euler–Lagrange equations (4.22) prove to be put in the vector form,

$$\frac{\mathrm{d}}{\mathrm{d}t}\dot{\zeta}^\alpha + \frac{\partial V_1}{\partial \zeta^\alpha} = \zeta^\alpha|\widetilde{\boldsymbol{\Omega}} \times \boldsymbol{e}_3|^2, \tag{4.27a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho(\zeta)\widetilde{\boldsymbol{\Omega}} \times \boldsymbol{e}_3) = P((\rho(\zeta)\widetilde{\boldsymbol{\Omega}} \times \boldsymbol{e}_3) \times \widetilde{\boldsymbol{\Omega}}). \tag{4.27b}$$

If we adopt local coordinates $(\zeta^\alpha, \theta, \phi, \dot{\zeta}^\alpha, \dot{\theta}, \dot{\phi})$ to express the Lagrangian, then equation (4.27*b*) can be shown to be equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\phi}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \phi} = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \theta} = 0. \tag{4.28}$$

The conserved quantity $g\big[\widetilde{P}\big(\frac{\partial L^*}{\partial \widetilde{\Pi}}\big), R(\boldsymbol{e}_3)\big]g^{-1}$ is written out as

$$g\left[\widetilde{P}\left(\frac{\partial L^*}{\partial \widetilde{\Pi}}\right), R(\boldsymbol{e}_3)\right]g^{-1} = -\frac{1}{2}R(g\widetilde{\boldsymbol{M}}), \tag{4.29}$$

where we have set

$$\widetilde{\boldsymbol{M}} := \rho(\zeta)P(\widetilde{\boldsymbol{\Omega}}) = \rho(\zeta)\boldsymbol{e}_3 \times (\widetilde{\boldsymbol{\Omega}} \times \boldsymbol{e}_3). \tag{4.30}$$

We note here that $g\widetilde{\boldsymbol{M}}$ and $\widetilde{\boldsymbol{M}}$ are the total angular momentum in the space frame and in the body frame, respectively;

$$g\widetilde{\boldsymbol{M}} = g\rho(\zeta)(\boldsymbol{e}_3 \times (\widetilde{\boldsymbol{\Omega}} \times \boldsymbol{e}_3)) = \rho(\zeta)\boldsymbol{u} \times \dot{\boldsymbol{u}}. \tag{4.31}$$

5. Hamiltonian mechanics for non-singular configurations

In this section, we derive the equations of motion for the non-singular configurations on the variational principle in the Hamiltonian formalism.

In section 3, we have taken $(q, g, \dot{q}, \Pi)$ as local coordinates in $T\dot{M}$. Given a Lagrangian $L(q, g, \dot{q}, \Pi)$, we define the generalized momenta $(p, \mathcal{M})$ conjugate to the $(\dot{q}, \Pi)$ to be

$$p_\alpha = \frac{\partial L}{\partial \dot{q}^\alpha}, \qquad \mathcal{M} = 2\frac{\partial L}{\partial \Pi}, \tag{5.1}$$

respectively. Note here that the factor 2 in the definition of $\mathcal{M}$ is necessary, because this definition gives rise to

$$\mathcal{M} = R(M) \qquad \text{with} \quad \boldsymbol{M} := \frac{\partial L}{\partial \boldsymbol{\pi}}. \tag{5.2}$$

We take $(q, g, p, \mathcal{M})$ as local coordinates in $T^*\dot{M}$. Then, the Hamiltonian $H(q, g, p, \mathcal{M})$ is given by

$$H(q, g, p, \mathcal{M}) = \sum_\alpha p_\alpha \dot{q}^\alpha + \frac{1}{2}\langle \mathcal{M}, \Pi\rangle - L(q, g, \dot{q}, \Pi), \tag{5.3}$$

where we note that

$$\frac{1}{2}\langle \mathcal{M}, \Pi\rangle = \boldsymbol{M} \cdot \boldsymbol{\pi}. \tag{5.4}$$

We are going to obtain the equations of motion for H on the basis of the variational principle applied to

$$\int_{t_1}^{t_2} \left(\sum_\alpha p_\alpha \frac{\mathrm{d}q^\alpha}{\mathrm{d}t} + \frac{1}{2}\langle \mathcal{M}, \Pi\rangle - H(q, g, p, \mathcal{M}) \right) \mathrm{d}t \tag{5.5}$$

with the boundary conditions

$$\delta q(t_i) = 0, \qquad \delta g(t_i) = 0, \qquad \delta p(t_i) = 0, \qquad \delta\mathcal{M}(t_i) = 0, \qquad i = 1, 2. \tag{5.6}$$

We note here that Hamilton's equations can be obtained without the boundary conditions for δp and $\delta\mathcal{M}$ at $t = t_i$. These boundary conditions are added only for further development of the theory in the Hamiltonian formalism. In a similar manner to that in the Lagrangian formalism, Hamilton's equations of motion are obtained as follows:

$$\dot{q}^\alpha = \frac{\partial H}{\partial p_\alpha}, \qquad \Pi = 2\frac{\partial H}{\partial \mathcal{M}}, \tag{5.7a}$$

$$\dot{p}_\alpha + \frac{\partial H}{\partial q^\alpha} = \frac{1}{2}\left\langle \mathcal{M}, \sum_\beta K_{\alpha\beta}\frac{\partial H}{\partial p_\beta}\right\rangle - \frac{1}{2}\left\langle \mathcal{M}, \left[2\frac{\partial H}{\partial \mathcal{M}}, \Lambda_\alpha\right]\right\rangle + \frac{1}{2}\left\langle g^{-1}\frac{\partial H}{\partial g} - \left(\frac{\partial H}{\partial g}\right)^T g, \Lambda_\alpha\right\rangle, \tag{5.7b}$$

$$\dot{\mathcal{M}} = \left[\mathcal{M}, 2\frac{\partial H}{\partial \mathcal{M}}\right] - \sum_\alpha [\mathcal{M}, \Lambda_\alpha]\frac{\partial H}{\partial p_\alpha} - \left(g^{-1}\frac{\partial H}{\partial g} - \left(\frac{\partial H}{\partial g}\right)^T g \right). \tag{5.7c}$$

Like (3.12), Hamilton's equation of motion (5.7) can be proved to be independent of the choice of local sections $U \to \dot{M}$.

Proposition 5.1. *The equations of motion for the non-singular configurations in the Hamiltonian formalism are expressed as (5.7), which are independent of the choice of local sections* $U \to \dot{M}$.

Assume that H is invariant under the left $SO(3)$ action,

$$H(q, hg, p, \mathcal{M}) = H(q, g, p, \mathcal{M}) \qquad \text{for all } h \in SO(3). \tag{5.8}$$

Note that Π is left-invariant and so is $\mathcal{M}$. Then, for $h = \mathrm{e}^{t\eta}$ with $\eta \in so(3)$, equation (5.8) is differentiated with respect to t at $t = 0$ to provide

$$\frac{\mathrm{d}}{\mathrm{d}t} H(q, \mathrm{e}^{t\eta} g, p, \mathcal{M})\bigg|_{t=0} = \left\langle \frac{\partial H}{\partial g}, \eta g \right\rangle = \frac{1}{2} \left\langle \frac{\partial H}{\partial g} g^{-1} - g \left(\frac{\partial H}{\partial g} \right)^T, \eta \right\rangle = 0. \tag{5.9}$$

Since $\eta \in so(3)$ is arbitrary, one obtains $\frac{\partial H}{\partial g} g^{-1} - g \left(\frac{\partial H}{\partial g} \right)^T = 0$, so that

$$g^{-1} \frac{\partial H}{\partial g} - \left(\frac{\partial H}{\partial g} \right)^T g = 0. \tag{5.10}$$

From the rotational invariance, the H will reduce to a function $H^*(q, p, \mathcal{M})$ on

$$T^* \dot{M}/G \cong T^*(\dot{M}/G) \oplus \widetilde{\mathcal{G}}^*, \tag{5.11}$$

where the right-hand side is a Whitney sum bundle, and $\widetilde{\mathcal{G}}^*$ denotes the covector bundle associated with the co-adjoint action of $SO(3)$ on $\mathcal{G}^* \cong so(3)$, $\widetilde{\mathcal{G}}^* := \dot{M} \times_G \mathcal{G}^*$ (see [14, 18] for cotangent bundle reduction). From (5.7) and (5.10), the reduced equations of motion are described as

$$\dot{q}^\alpha = \frac{\partial H^*}{\partial p_\alpha}, \qquad \Pi = 2 \frac{\partial H^*}{\partial \mathcal{M}}, \tag{5.12a}$$

$$\dot{p}_\alpha + \frac{\partial H^*}{\partial q^\alpha} = \frac{1}{2} \left\langle \mathcal{M}, \sum_\beta K_{\alpha\beta} \frac{\partial H^*}{\partial p_\beta} \right\rangle - \frac{1}{2} \left\langle \mathcal{M}, \left[2 \frac{\partial H^*}{\partial \mathcal{M}}, \Lambda_\alpha \right] \right\rangle, \tag{5.12b}$$

$$\dot{\mathcal{M}} = \left[\mathcal{M}, 2 \frac{\partial H^*}{\partial \mathcal{M}} \right] - \sum_\alpha [\mathcal{M}, \Lambda_\alpha] \frac{\partial H^*}{\partial p_\alpha}. \tag{5.12c}$$

From (5.12c), which is expressed also as $\dot{\mathcal{M}} = [\mathcal{M}, \xi]$, it follows that the quantity $g \mathcal{M} g^{-1}$ is conserved. In fact, one easily verifies that

$$\frac{\mathrm{d}}{\mathrm{d}t} (g \mathcal{M} g^{-1}) = 0. \tag{5.13}$$

Hence, $\mathcal{M}$ is put in the form $\mathcal{M} = g^{-1} \zeta g$ with a constant $\zeta \in so(3)^* \cong so(3)$. This implies that $\mathcal{M}$ is tracking on a coadjoint orbit in each fibre of $\widetilde{\mathcal{G}}^*$. According to [14], the symplectic leaves of $T^* \dot{M}/G$ are given by $\mathbb{J}^{-1}(\mathcal{O})/G$ for each coadjoint orbit $\mathcal{O}$ in $\mathcal{G}^* \cong so(3)$, where $\mathbb{J} = g \mathcal{M} g^{-1}$ in the present case. Further, $\mathbb{J}^{-1}(\mathcal{O})/G$ is canonically diffeomorphic to the reduced phase space $\mathbb{J}^{-1}(\mu)/G_\mu$, where $\mu \in \mathcal{O}$ and where G_μ denotes the isotropy subgroup at μ [18].

Proposition 5.2. *If the Hamiltonian is rotationally invariant, the equations of motion for the non-singular configurations in the Hamiltonian formalism reduce to (5.12). Further, the quantity* $g \mathcal{M} g^{-1}$ *is conserved.*

We now derive the equations of motion for an N-particle system. Given the Lagrangian (3.24), we obtain, by (5.3), the Hamiltonian

$$H^* = \frac{1}{2}\sum_{\alpha,\beta} a^{\alpha\beta} p_\alpha p_\beta + \frac{1}{4}\langle \mathcal{M}, \widetilde{A}^{-1}\mathcal{M}\rangle + V(q), \qquad \widetilde{A}^{-1} = RA^{-1}R^{-1}. \tag{5.14}$$

Note here that

$$p_\alpha = \sum_\beta a_{\alpha\beta}\dot{q}^\beta, \qquad \mathcal{M} = \widetilde{A}\Pi. \tag{5.15}$$

Then the reduced equations of motion (5.12) are equivalently written in vector notation as

$$\dot{q}^\alpha = \sum_\beta a^{\alpha\beta} p_\beta, \qquad \boldsymbol{\pi} = A^{-1}\boldsymbol{M}, \tag{5.16a}$$

$$\dot{p}_\alpha - \sum_{\beta,\gamma,\mu} a^{\beta\mu}\Gamma^\gamma_{\mu\alpha} p_\gamma p_\beta + \frac{\partial V}{\partial q^\alpha} = \sum_{\beta,\gamma} \boldsymbol{M}\cdot\boldsymbol{\kappa}_{\alpha\beta} a^{\beta\gamma} p_\gamma - \frac{1}{2}\boldsymbol{M}\cdot\left(\frac{\partial A^{-1}}{\partial q^\alpha} - [\Lambda_\alpha, A^{-1}]\right)\boldsymbol{M}, \tag{5.16b}$$

$$\frac{\mathrm{d}\boldsymbol{M}}{\mathrm{d}t} - \sum_{\alpha,\beta} a^{\alpha\beta} p_\alpha \boldsymbol{\lambda}_\beta \times \boldsymbol{M} = \boldsymbol{M}\times A^{-1}\boldsymbol{M}, \tag{5.16c}$$

where $\boldsymbol{\kappa}_{\alpha\beta}$ and $\boldsymbol{\lambda}_\alpha$ were defined in (3.26). Since $g\mathcal{M}g^{-1} = R(g\boldsymbol{M})$, the total angular momentum $\boldsymbol{L} = g\boldsymbol{M}$ is conserved. The quantity $\frac{\partial A^{-1}}{\partial q^\alpha} - [\Lambda_\alpha, A^{-1}]$ is viewed as the covariant derivative of A^{-1}.

To compare Hamilton's equations of motion with the Euler–Lagrange equations, the following equation is of great help:

$$\frac{\mathrm{D}A^{-1}}{\partial q^\alpha}\boldsymbol{M}\cdot\boldsymbol{M} + \boldsymbol{\pi}\cdot\frac{\mathrm{D}A}{\partial q^\alpha}\boldsymbol{\pi} = 0, \tag{5.17}$$

which can be proved by the use of the fact that $\Lambda_\alpha A + A\Lambda_\alpha$ is an anti-symmetric matrix. It is easy to show that (5.16b) with the first equation of (5.16a) are equivalent to (3.25a) and that (5.16c) with the second equation of (5.16a) is equivalent to (3.25b).

In conclusion of this section, we make a comment on momentum variables. If we had started with the Lagrangian of the form (3.33), we would have obtained the momentum variable in the form

$$p_\alpha = \frac{\partial L^*}{\partial \dot{q}^\alpha} = \sum_\beta a_{\alpha\beta}\dot{q}^\beta + \boldsymbol{\lambda}_\alpha\cdot A_{\sigma(q)}\left(\boldsymbol{\Omega} + \sum_\beta \boldsymbol{\lambda}_\beta\dot{q}^\beta\right), \tag{5.18}$$

which is usually adopted in the physics literature [7].

6. Hamiltonian mechanics for collinear configurations

In this section, we derive the equations of motion for collinear configurations on the variational principle in the Hamiltonian formalism.

We have taken $(\zeta, \boldsymbol{u}, \dot{\zeta}, \widetilde{\Pi})$ as local coordinates in TM_1 in section 4. Let $L(\zeta, \boldsymbol{u}, \dot{\zeta}, \widetilde{\Pi})$ be a Lagrangian on TM_1. We define the generalized momenta $(\varpi, \widetilde{\mathcal{M}})$ conjugate to the $(\dot{\zeta}, \widetilde{\Pi})$ to be

$$\varpi_\alpha = \frac{\partial L}{\partial \dot{\zeta}^\alpha}, \qquad \widetilde{\mathcal{M}} = 2\frac{\partial L}{\partial \widetilde{\Pi}}, \tag{6.1}$$

respectively. We now take $(\zeta, \boldsymbol{u}, \varpi, \widetilde{\mathcal{M}})$ as local coordinates in T^*M_1. Then the Hamiltonian $H(\zeta, \boldsymbol{u}, \varpi, \widetilde{\mathcal{M}})$ is given by

$$H(\zeta, \boldsymbol{u}, \varpi, \widetilde{\mathcal{M}}) = \sum_\alpha \varpi_\alpha \dot{\zeta}^\alpha + \frac{1}{2}\langle \widetilde{\mathcal{M}}, \widetilde{\Pi}\rangle - L(\zeta, \boldsymbol{u}, \dot{\zeta}, \widetilde{\Pi}). \tag{6.2}$$

The equations of motion for H will be obtained on the basis of the variational principle applied to

$$\int_{t_1}^{t_2} \left(\sum_\alpha \varpi_\alpha \frac{\mathrm{d}\zeta^\alpha}{\mathrm{d}t} + \frac{1}{2}\langle \widetilde{\mathcal{M}}, \widetilde{\Pi}\rangle - H(\zeta, \boldsymbol{u}, \varpi, \widetilde{\mathcal{M}}) \right) \mathrm{d}t \tag{6.3}$$

with the boundary conditions

$$\delta\zeta(t_i) = 0, \qquad \delta\tilde{g}(t_i) = 0, \qquad \delta\varpi(t_i) = 0, \qquad \delta\tilde{\mathcal{M}}(t_i) = 0, \qquad i = 1, 2. \tag{6.4}$$

It then turns out that Hamiltonian equations of motion are put in the form

$$\dot{\zeta}^\alpha = \frac{\partial H}{\partial \varpi_\alpha}, \qquad \widetilde{\Pi} = 2\frac{\partial H}{\partial \widetilde{\mathcal{M}}}, \tag{6.5a}$$

$$\dot{\varpi}_\alpha = -\frac{\partial H}{\partial \zeta^\alpha}, \tag{6.5b}$$

$$\widetilde{P}\left(\dot{\widetilde{\mathcal{M}}} - [\widetilde{P}(\widetilde{\mathcal{M}}), \widetilde{\xi}] + R\left(\widetilde{g}^{-1}\frac{\partial H}{\partial \boldsymbol{u}} \right) \right) = 0, \tag{6.5c}$$

$$\left[\widetilde{P}(\widetilde{\mathcal{M}}), 2\frac{\partial H}{\partial \widetilde{\mathcal{M}}} \right] = 0, \tag{6.5d}$$

where (6.5*d*) is a constraint to be required.

Proposition 6.1. *The equations of motion for collinear configurations in the Hamiltonian formalism are expressed as (6.5), which are independent of the choice of local sections* $\widetilde{U} \to M_1$.

We now assume that H is rotationally invariant,

$$H(\zeta, \mathrm{e}^{tR(\boldsymbol{a})}\boldsymbol{u}, \varpi, \widetilde{\mathcal{M}}) = H(\zeta, \boldsymbol{u}, \varpi, \widetilde{\mathcal{M}}) \qquad \text{for all } R(\boldsymbol{a}) \in so(3). \tag{6.6}$$

Note that $\widetilde{\Pi}$ is left-invariant and so is $\widetilde{\mathcal{M}}$. From (6.6), it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t} H(\zeta, \mathrm{e}^{tR(\boldsymbol{a})}\boldsymbol{u}, \varpi, \widetilde{\mathcal{M}})\bigg|_{t=0} = R(\boldsymbol{a})\boldsymbol{u} \cdot \frac{\partial H}{\partial \boldsymbol{u}} = \boldsymbol{a} \cdot \left(\boldsymbol{u} \times \frac{\partial H}{\partial \boldsymbol{u}} \right) = 0, \tag{6.7}$$

so that

$$\boldsymbol{u} \times \frac{\partial H}{\partial \boldsymbol{u}} = \widetilde{g}\left(\boldsymbol{e}_3 \times \widetilde{g}^{-1}\frac{\partial H}{\partial \boldsymbol{u}} \right) = 0. \tag{6.8}$$

This implies that

$$P\left(\widetilde{g}^{-1}\frac{\partial H}{\partial \boldsymbol{u}} \right) = 0. \tag{6.9}$$

From (6.6), H will reduce to a function $H^*(\zeta, \varpi, \widetilde{\Pi})$ on

$$T^*(M_1/SO(3)) \oplus \widetilde{\mathcal{G}}_1^*, \tag{6.10}$$

where $\widetilde{\mathcal{G}}_1^*$ is a covector bundle over $M_1/SO(3)$. From (6.9), Hamilton's equations of motion (6.5) reduce to

$$\dot{\zeta}^\alpha = \frac{\partial H^*}{\partial \varpi_\alpha}, \qquad \widetilde{\Pi} = 2\frac{\partial H^*}{\partial \widetilde{\mathcal{M}}}, \tag{6.11a}$$

$$\dot{\varpi}_\alpha = -\frac{\partial H^*}{\partial \zeta^\alpha}, \qquad \widetilde{P}(\dot{\widetilde{\mathcal{M}}} - [\widetilde{P}(\widetilde{\mathcal{M}}), \widetilde{\xi}]) = 0, \tag{6.11b}$$

$$\left[\widetilde{P}(\widetilde{\mathcal{M}}), 2\frac{\partial H^*}{\partial \widetilde{\mathcal{M}}}\right] = 0. \tag{6.11c}$$

Proposition 6.2. *If the Hamiltonian is rotationally invariant, Hamilton's equations of motion for collinear configurations reduce to (6.11), which are defined on the reduced bundle (6.10).*

Given Lagrangian (4.25), we obtain, from (6.2), the Hamiltonian

$$H^* = \frac{1}{2}\sum_\alpha \varpi_\alpha^2 + \frac{1}{4\rho(\zeta)}\langle \widetilde{\mathcal{M}}, \widetilde{\mathcal{M}}\rangle + V_1(\zeta). \tag{6.12}$$

Since $\widetilde{\mathcal{M}} = \rho(\zeta)\widetilde{\Pi}$, and hence $\widetilde{P}(\widetilde{\mathcal{M}}) = \widetilde{\mathcal{M}}$, equation (6.11$c$) is satisfied. The reduced equations of motion are then put in vector form

$$\dot{\zeta}^\alpha = \varpi_\alpha, \qquad \widetilde{\boldsymbol{\Omega}} \times \boldsymbol{e}_3 = \frac{1}{\rho(\zeta)}\widetilde{\boldsymbol{M}}, \tag{6.13a}$$

$$\dot{\varpi}_\alpha = \frac{\zeta^\alpha}{\rho(\zeta)^2}|\widetilde{\boldsymbol{M}}|^2 - \frac{\partial V_1}{\partial \zeta^\alpha}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\boldsymbol{M}} = P(\widetilde{\boldsymbol{M}} \times \widetilde{\boldsymbol{\Omega}}), \tag{6.13b}$$

where $\widetilde{\mathcal{M}} = R(\widetilde{\boldsymbol{M}})$. Note that equations (6.13) are equivalent to (4.27).

7. Three-body systems

In this section, we wish to study the Euler–Lagrange equations around collinear configurations. For simplicity, we specialize in three-body systems. In the same manner as in quantum systems [9], we introduce internal coordinates (q_1, q_2, q_3) by

$$q_1 = r_1, \qquad q_2 = r_2\cos\varphi, \qquad q_3 = r_2\sin\varphi, \tag{7.1}$$

where

$$r_1 = \|\boldsymbol{r}_1\|, \qquad r_2 = \|\boldsymbol{r}_2\|, \qquad \boldsymbol{r}_1 \cdot \boldsymbol{r}_2 = r_1 r_2 \cos\varphi, \tag{7.2}$$

and define a local section $\sigma(q) = (\boldsymbol{\sigma}_1(q), \boldsymbol{\sigma}_2(q))$ by

$$\boldsymbol{\sigma}_1(q) = q_1\boldsymbol{e}_3, \qquad \boldsymbol{\sigma}_2(q) = q_2\boldsymbol{e}_3 + q_3\boldsymbol{e}_1. \tag{7.3}$$

We note here that subscript indices are used in describing local coordinates. Though the local section σ is defined originally on an open subset U of $\dot{Q} = \dot{M}/SO(3)$, i.e., two shape coordinates q_1 and q_3 on the open subset U must be restricted to positive numbers, we can take (q_1, q_2, q_3) as local coordinates beyond U,

$$\{(q_1, q_2, q_3) | q_1 \geqslant 0, q_3 \geqslant 0\}. \tag{7.4}$$

Then we have collinear configurations if $q_3 = 0$, and the configurations that two of three particles collide but the remainder is separate, if $q_1 = 0$. If $q_1 = q_2 = q_3 = 0$, we have a triple collision.

Let $g \in SO(3)$ be expressed as $g = e^{\phi R(e_3)} e^{\theta R(e_2)} e^{\psi R(e_3)}$ in terms of the Euler angles. Then, from (7.3), the Jacobi vectors are expressed as

$$g\boldsymbol{\sigma}_1(q) = e^{\phi R(\boldsymbol{e}_3)} e^{\theta R(\boldsymbol{e}_2)} q_1 \boldsymbol{e}_3 \tag{7.5a}$$

$$g\boldsymbol{\sigma}_2(q) = e^{\phi R(\boldsymbol{e}_3)} e^{\theta R(\boldsymbol{e}_2)} (q_2 \boldsymbol{e}_3 + q_3 \cos\psi \boldsymbol{e}_1 + q_3 \sin\psi \boldsymbol{e}_2). \tag{7.5b}$$

If $q_3 \to 0$, the local coordinates of $\dot{M}$ reduce to those of M_1; $(q_1, q_2, q_3, \phi, \theta, \psi) \to (q_1, q_2, \phi, \theta)$.

From definition (2.12) along with (7.3), the inertia tensor and its inverse at $\sigma(q)$ are put, respectively, in the form

$$A_{\sigma(q)} = \begin{pmatrix} q_1^2 + q_2^2 & 0 & -q_2 q_3 \\ 0 & q_1^2 + q_2^2 + q_3^2 & 0 \\ -q_2 q_3 & 0 & q_3^2 \end{pmatrix}, \tag{7.6}$$

$$A_{\sigma(q)}^{-1} = \begin{pmatrix} \frac{1}{q_1^2} & 0 & \frac{q_2}{q_1^2 q_3} \\ 0 & \frac{1}{q_1^2+q_2^2+q_3^2} & 0 \\ \frac{q_2}{q_1^2 q_3} & 0 & \frac{q_1^2+q_2^2}{q_1^2 q_3^2} \end{pmatrix}. \tag{7.7}$$

From (2.19), (7.3) and (7.7), the connection form (2.19) proves to be expressed as

$$\omega_{\sigma(q)} = \frac{q_2\,dq_3 - q_3\,dq_2}{q_1^2 + q_2^2 + q_3^2} R(\boldsymbol{e}_2). \tag{7.8}$$

Thus, the vectors, $\boldsymbol{\lambda}_\alpha$, $\alpha = 1, 2, 3$ associated with $\omega_{\sigma(q)} = \sum_\alpha R(\boldsymbol{\lambda}_\alpha)\,dq_\alpha$ are put in the form

$$\boldsymbol{\lambda}_1 = \boldsymbol{0}, \qquad \boldsymbol{\lambda}_2 = -\frac{q_3}{q_1^2 + q_2^2 + q_3^2}\boldsymbol{e}_2, \qquad \boldsymbol{\lambda}_3 = \frac{q_2}{q_1^2 + q_2^2 + q_3^2}\boldsymbol{e}_2, \tag{7.9}$$

respectively. From this and (3.1)–(3.3), it follows that

$$\boldsymbol{\pi} = \boldsymbol{\Omega} + \frac{q_2\dot{q}_3 - q_3\dot{q}_2}{q_1^2 + q_2^2 + q_3^2}\boldsymbol{e}_2. \tag{7.10}$$

From (3.13) and (7.9), the curvature tensors $\boldsymbol{\kappa}_{\alpha\beta}$, $\alpha, \beta = 1, 2, 3$, are calculated as

$$\begin{aligned} &\boldsymbol{\kappa}_{11} = \boldsymbol{\kappa}_{22} = \boldsymbol{\kappa}_{33} = \boldsymbol{0}, \qquad \boldsymbol{\kappa}_{12} = -\boldsymbol{\kappa}_{21} = \frac{2q_1 q_3}{(q_1^2 + q_2^2 + q_3^2)^2}\boldsymbol{e}_2, \\ &\boldsymbol{\kappa}_{23} = -\boldsymbol{\kappa}_{32} = \frac{2q_1^2}{(q_1^2 + q_2^2 + q_3^2)^2}\boldsymbol{e}_2, \qquad \boldsymbol{\kappa}_{31} = -\boldsymbol{\kappa}_{13} = \frac{2q_1 q_2}{(q_1^2 + q_2^2 + q_3^2)^2}\boldsymbol{e}_2. \end{aligned} \tag{7.11}$$

The metric tensor and its inverse are expressed, respectively, as

$$(a_{\alpha\beta}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{q_1^2+q_2^2}{q_1^2+q_2^2+q_3^2} & \frac{q_2 q_3}{q_1^2+q_2^2+q_3^2} \\ 0 & \frac{q_2 q_3}{q_1^2+q_2^2+q_3^2} & \frac{q_1^2+q_3^2}{q_1^2+q_2^2+q_3^2} \end{pmatrix}, \tag{7.12}$$

$$(a^{\alpha\beta}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{q_1^2+q_3^2}{q_1^2} & -\frac{q_2 q_3}{q_1^2} \\ 0 & -\frac{q_2 q_3}{q_1^2} & \frac{q_1^2+q_2^2}{q_1^2} \end{pmatrix}. \tag{7.13}$$

See [9] for details. Then the Christoffel symbols $\Gamma^{\alpha}_{\beta\gamma}$ are calculated, from (3.27) with (7.12), (7.13) as

$$\begin{aligned}
&\Gamma^1_{22} = -\frac{q_1 q_3^2}{\left(q_1^2+q_2^2+q_3^2\right)^2}, \qquad \Gamma^1_{33} = -\frac{q_1 q_2^2}{\left(q_1^2+q_2^2+q_3^2\right)^2},\\
&\Gamma^1_{23} = \Gamma^1_{32} = \frac{q_1 q_2 q_3}{\left(q_1^2+q_2^2+q_3^2\right)^2}, \qquad \Gamma^2_{22} = -\frac{q_2 q_3^2}{\left(q_1^2+q_2^2+q_3^2\right)^2},\\
&\Gamma^2_{33} = \frac{q_2\left(2q_1^2+q_2^2+2q_3^2\right)}{\left(q_1^2+q_2^2+q_3^2\right)^2}, \qquad \Gamma^2_{12} = \Gamma^2_{21} = \frac{q_3^2}{q_1\left(q_1^2+q_2^2+q_3^2\right)},\\
&\Gamma^2_{13} = \Gamma^2_{31} = -\frac{q_2 q_3}{q_1\left(q_1^2+q_2^2+q_3^2\right)}, \qquad \Gamma^2_{23} = \Gamma^2_{32} = -\frac{q_3\left(q_1^2+q_3^2\right)}{\left(q_1^2+q_2^2+q_3^2\right)^2},\\
&\Gamma^3_{22} = \frac{q_3\left(2q_1^2+2q_2^2+q_3^2\right)}{\left(q_1^2+q_2^2+q_3^2\right)^2}, \qquad \Gamma^3_{33} = -\frac{q_2^2 q_3}{\left(q_1^2+q_2^2+q_3^2\right)^2},\\
&\Gamma^3_{12} = \Gamma^3_{21} = -\frac{q_2 q_3}{q_1\left(q_1^2+q_2^2+q_3^2\right)}, \qquad \Gamma^3_{13} = \Gamma^3_{31} = \frac{q_2^2}{q_1\left(q_1^2+q_2^2+q_3^2\right)},\\
&\Gamma^3_{23} = \Gamma^3_{32} = -\frac{q_2\left(q_1^2+q_2^2\right)}{\left(q_1^2+q_2^2+q_3^2\right)^2}, \qquad \Gamma^{\alpha}_{\beta\gamma} = 0 \quad \text{(otherwise)}.
\end{aligned} \tag{7.14}$$

Thus, the vibrational part (3.25a) of the reduced Euler–Lagrange equations for the three-body system is written out as

$$\frac{\mathrm{d}}{\mathrm{d}t}\dot{q}_1 - \frac{q_1(q_3\dot{q}_2 - q_2\dot{q}_3)^2}{\left(q_1^2+q_2^2+q_3^2\right)^2} + \frac{\partial V}{\partial q_1} = q_1\left(\pi_1^2+\pi_2^2\right) + \frac{2q_1(q_3\dot{q}_2 - q_2\dot{q}_3)}{q_1^2+q_2^2+q_3^2}\pi_2, \tag{7.15a}$$

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\dot{q}_2 - \frac{q_2\dot{q}_3 - q_3\dot{q}_2}{q_1^2+q_2^2+q_3^2}\left(\frac{2(q_3\dot{q}_1 - q_1\dot{q}_3)}{q_1} + \frac{q_2(q_2\dot{q}_3 - q_3\dot{q}_2)}{q_1^2+q_2^2+q_3^2}\right) + \frac{q_1^2+q_3^2}{q_1^2}\frac{\partial V}{\partial q_2} - \frac{q_2 q_3}{q_1^2}\frac{\partial V}{\partial q_3}\\
&\quad = q_2\left(\pi_1^2+\pi_2^2\right) + \frac{2q_3}{q_1^2}\left(q_2 q_3\left(\pi_1^2-\pi_3^2\right) + \left(q_2^2-q_3^2\right)\pi_1\pi_3\right)\\
&\qquad - 2\pi_2\left(\frac{q_3\dot{q}_1 - q_1\dot{q}_3}{q_1} + \frac{q_2(q_2\dot{q}_3 - q_3\dot{q}_2)}{q_1^2+q_2^2+q_3^2}\right),
\end{aligned} \tag{7.15b}$$

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\dot{q}_3 - \frac{q_2\dot{q}_3 - q_3\dot{q}_2}{q_1^2+q_2^2+q_3^2}\left(\frac{2(q_1\dot{q}_2 - q_2\dot{q}_1)}{q_1} + \frac{q_3(q_2\dot{q}_3 - q_3\dot{q}_2)}{q_1^2+q_2^2+q_3^2}\right) - \frac{q_2 q_3}{q_1^2}\frac{\partial V}{\partial q_2} + \frac{q_1^2+q_2^2}{q_1^2}\frac{\partial V}{\partial q_3}\\
&\quad = q_3\left(\pi_2^2+\pi_3^2\right) - \frac{2q_2}{q_1^2}\left(q_2 q_3\left(\pi_1^2-\pi_3^2\right) + \left(q_1^2+q_2^2-q_3^2\right)\pi_1\pi_3\right)\\
&\qquad - 2\pi_2\left(\frac{q_1\dot{q}_2 - q_2\dot{q}_1}{q_1} + \frac{q_3(q_2\dot{q}_3 - q_3\dot{q}_2)}{q_1^2+q_2^2+q_3^2}\right).
\end{aligned} \tag{7.15c}$$

The rotational part (3.25*b*) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left(q_1^2+q_2^2\right)\begin{pmatrix}\Omega_1\\ \Omega_2\\ 0\end{pmatrix}\right)+\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}-q_2q_3\Omega_3\\ q_3^2\Omega_2+q_2\dot{q}_3-q_3\dot{q}_2\\ -q_2q_3\Omega_1+q_3^2\Omega_3\end{pmatrix}$$
$$=\left(q_1^2+q_2^2\right)\begin{pmatrix}\Omega_2\Omega_3\\ -\Omega_1\Omega_3\\ 0\end{pmatrix}+q_3^2\begin{pmatrix}0\\ \Omega_1\Omega_3\\ -\Omega_1\Omega_2\end{pmatrix}$$
$$+q_2q_3\begin{pmatrix}\Omega_1\Omega_2\\ \Omega_1^2+\Omega_3^2\\ -\Omega_2\Omega_3\end{pmatrix}+(q_2\dot{q}_3-q_3\dot{q}_2)\begin{pmatrix}\Omega_3\\ 0\\ -\Omega_1\end{pmatrix}. \tag{7.16}$$

Assume now that equation (7.16) is compatible with the collinear constraint $q_3 = 0$. Then it reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left(q_1^2+q_2^2\right)\begin{pmatrix}\Omega_1\\ \Omega_2\\ 0\end{pmatrix}\right)=\left(q_1^2+q_2^2\right)\begin{pmatrix}\Omega_2\Omega_3\\ -\Omega_1\Omega_3\\ 0\end{pmatrix}, \tag{7.17}$$

which is equivalent to the rotational part (4.27*b*) of the Euler–Lagrange equations for the collinear configurations. Note here that $\boldsymbol{\Omega}$ reduces to $\widetilde{\boldsymbol{\Omega}}$ as q_3 tends to $q_3 = 0$.

We now try to impose the constraint $q_3 = 0$ on the vibrational part (7.15) of the Euler–Lagrange equations for non-singular configurations. From (7.15) with $q_3 = 0$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\dot{q}_1+\frac{\partial V}{\partial q_1}=q_1\left(\pi_1^2+\pi_2^2\right), \tag{7.18a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\dot{q}_2+\frac{\partial V}{\partial q_2}=q_2\left(\pi_1^2+\pi_2^2\right), \tag{7.18b}$$

$$\frac{q_1^2+q_2^2}{q_1^2}\frac{\partial V}{\partial q_3}=-\frac{2q_2\left(q_1^2+q_2^2\right)}{q_1^2}\pi_1\pi_3-\frac{2(q_1\dot{q}_2-q_2\dot{q}_1)}{q_1}\pi_2. \tag{7.18c}$$

Since $\boldsymbol{\pi} = \boldsymbol{\Omega}$ if $q_3 = 0$, equations (7.18*a*) and (7.18*b*) coincide with equation (4.27*a*) for collinear configurations. If the constraints $\pi_3 = 0$ and $q_1\dot{q}_2 - q_2\dot{q}_1 = 0$ are satisfied furthermore, and if the potential V has the property such that $\frac{\partial V}{\partial q_3} = 0$ at $q_3 = 0$, equation (7.18*c*) holds true. The constraint $\pi_3(=\Omega_3) = 0$ means that the angular momentum has a vanishing component around the axis of three-body alignment. The constraint $q_1\dot{q}_2 - q_2\dot{q}_1 = 0$ implies that $\dot{\boldsymbol{r}}_1 // \dot{\boldsymbol{r}}_2$. It then turns out that collinear motion can take place if the conditions $\Omega_3 = 0$ and $q_1\dot{q}_2 - q_2\dot{q}_1 = 0$ are satisfied as well as $q_3 = 0$, and if the potential V has the property such that $\frac{\partial V}{\partial q_3} = 0$ at $q_3 = 0$.

8. Remarks

In this section, we make remarks on the covariant derivatives which were used in describing the reduced equations of motion and on the Euler–Lagrange equations from the viewpoint of Hamel [22, 23].

8.1. Covariant derivatives

The group $SO(3)$ is represented in $\mathbf{R}^3$ in a natural manner. The product space $\dot{M} \times \mathbf{R}^3$ is endowed with the equivalence relation by $(x, v) \sim (gx, gv)$ with $g \in SO(3)$. The factor space $\dot{M} \times_{SO(3)} \mathbf{R}^3$ becomes a vector bundle over $\dot{M}/SO(3)$. A vector field $\boldsymbol{V}$ is called an equivariant vector field on $\dot{M}$, if it transforms according to $\boldsymbol{V}_{hx} = h\boldsymbol{V}_x$ for $h \in SO(3)$. The set of equivariant vector fields is in one-to-one correspondence with the set of sections in $\dot{M} \times_{SO(3)} \mathbf{R}^3$; $\boldsymbol{s}(\pi(x)) = [(x, \boldsymbol{V}_x)]$. We denote the correspondence by $\boldsymbol{s} = \gamma \boldsymbol{V}$. The covariant derivative of the local section $\boldsymbol{s}$ with respect to a vector field X on $\dot{M}/SO(3)$ is defined by

$$\nabla_X \boldsymbol{s} = \gamma X^*(\gamma^{-1}\boldsymbol{s}), \tag{8.1}$$

where X^* is the horizontal lift of X.

For a local section $\sigma : U \to \dot{M}$ and an equivariant vector field $\boldsymbol{V}$, one has $x = g\sigma(q)$, and hence

$$\boldsymbol{s}(\pi(x)) = [(x, \boldsymbol{V}_x)] = [(\sigma(q), \boldsymbol{V}_{\sigma(q)})]. \tag{8.2}$$

This implies that one can view $\boldsymbol{V}_{\sigma(q)}$ as a local section in the vector bundle $\dot{M} \times_{SO(3)} \mathbf{R}^3$, and that $\boldsymbol{V}_{\sigma(q)}$ transforms as $\boldsymbol{V}_{\sigma'(q)} = h\boldsymbol{V}_{\sigma(q)}$, where $\sigma'(q) = h(q)\sigma(q)$. We now calculate the covariant derivative of a local section $q \mapsto \boldsymbol{v}_q = \boldsymbol{V}_{\sigma(q)}$ with respect to $\frac{\partial}{\partial q^\alpha}$. For the equivariant vector field $\boldsymbol{V}$, its derivative with respect to K_a is given by

$$(K_a \boldsymbol{V})_x = \frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{V}_{g\exp(tR(\boldsymbol{e}_a))\sigma(q)}\bigg|_{t=0} = \mathrm{Ad}_g(R(\boldsymbol{e}_a))\boldsymbol{V}_x,$$

so that

$$\sum_a \Lambda^a_\alpha K_a \boldsymbol{V} = \mathrm{Ad}_g(\Lambda_\alpha)\boldsymbol{V}. \tag{8.3}$$

Then the horizontal derivative of $\boldsymbol{V}$ with respect to $\left(\frac{\partial}{\partial q^\alpha}\right)^*$ is expressed as

$$\left(\frac{\partial}{\partial q^\alpha}\right)^* \boldsymbol{V} = \left(\frac{\partial}{\partial q^\alpha} - \sum_\alpha \Lambda^a_\alpha K_a\right)\boldsymbol{V} = \mathrm{Ad}_g\left(\frac{\partial}{\partial q^\alpha} \otimes I - \Lambda_\alpha\right)\boldsymbol{V}. \tag{8.4}$$

Restricting $\left(\frac{\partial}{\partial q^\alpha}\right)^* \boldsymbol{V}$ on $\sigma(q)$, we obtain

$$(\nabla_{\partial/\partial q^\alpha}\boldsymbol{s})(q) = \left[\left(\sigma(q), \left(\left(\frac{\partial}{\partial q^\alpha} - \Lambda_\alpha\right)\boldsymbol{V}\right)_{\sigma(q)}\right)\right], \tag{8.5}$$

which implies that the covariant derivative of a local section $\boldsymbol{v}$ is given by

$$\left(\frac{\partial}{\partial q^\alpha} - \Lambda_\alpha\right)\boldsymbol{v}. \tag{8.6}$$

Note that the vector bundle $\dot{M} \times_{SO(3)} \mathbf{R}^3$ is identified with the adjoint bundle $\widetilde{\mathcal{G}} = \dot{M} \times_{SO(3)} \mathcal{G}$, which is defined through the equivalence relation $(x, \xi) \sim (gx, \mathrm{Ad}_g\xi)$.

Let $\mathrm{Sym}(3, \mathbf{R})$ denote the space of real symmetric 3×3 matrices, on which $SO(3)$ acts in the manner $S \mapsto gSg^{-1}$ for $g \in SO(3)$. A $\mathrm{Sym}(3, \mathbf{R})$-valued function T on $\dot{M}$ is called equivariant, if it transforms according to

$$T_{hx} = hT_xh^{-1}, \qquad h \in SO(3). \tag{8.7}$$

The set of $\mathrm{Sym}(3, \mathbf{R})$-valued equivariant functions is in one-to-one correspondence with the set of sections in the tensor bundle $\dot{M} \times_{SO(3)} \mathrm{Sym}(3, \mathbf{R})$. We denote the correspondence by $s = \gamma T$. Then the covariant derivative of the section s with respect to a vector field X on $\dot{M}/SO(3)$ is defined by

$$\nabla_X s = \gamma X^*(\gamma^{-1}s), \tag{8.8}$$

where X^* is the horizontal lift of X.

For a local section $\sigma : U \to \dot{M}$ and an equivariant function T, one has

$$s(\pi(x)) = [(x, T_x)] = [(\sigma(q), T_{\sigma(q)})], \tag{8.9}$$

which means that we may look on $T_{\sigma(q)}$ as a local section in $\dot{M} \times_{SO(3)} \mathrm{Sym}(3, \mathbf{R})$. For the equivariant function T, its derivative with respect to K_a is given by

$$(K_a T)_x = \left.\frac{\mathrm{d}}{\mathrm{d}t} T_{g\exp(tR(e_a))\sigma(q)}\right|_{t=0} = [\mathrm{Ad}_g R(e_a), T_x], \tag{8.10}$$

so that

$$\sum_a \Lambda_\alpha^a K_a T = [\mathrm{Ad}_g \Lambda_\alpha, T]. \tag{8.11}$$

Then the horizontal derivative of T is given by

$$\left(\frac{\partial}{\partial q^\alpha}\right)^* T = \frac{\partial T}{\partial q^\alpha} - [\mathrm{Ad}_g \Lambda_\alpha, T]. \tag{8.12}$$

Restricting $\left(\frac{\partial}{\partial q^\alpha}\right)^* T$ on $\sigma(q)$, we obtain

$$\nabla_{\partial/\partial q^\alpha} s(q) = \left[\left(\sigma(q), \left(\frac{\partial T}{\partial q^\alpha} - [\Lambda_\alpha, T]\right)_{\sigma(q)}\right)\right]. \tag{8.13}$$

Hence, the covariant derivative of the inertia tensor $A = A_{\sigma(q)}$, a local section in $\dot{M} \times_{SO(3)} \mathrm{Sym}(3, \mathbf{R})$, is given by

$$\frac{\mathrm{D}A}{\partial q^\alpha} = \frac{\partial A}{\partial q^\alpha} - [\Lambda_\alpha, A]. \tag{8.14}$$

8.2. Hamel's approach

We show that the Euler–Lagrange equations (3.12) can also be derived from the Euler–Lagrange equations of the usual form. Let θ^λ and X_λ be local bases of 1-forms and of vector fields on an open subset W of $\mathbf{R}^n$, which are given by

$$\theta^\lambda = \sum_\mu A_\mu^\lambda \,\mathrm{d}x^\mu, \qquad X_\lambda = \sum_\mu B_\lambda^\mu \frac{\partial}{\partial x^\mu} \tag{8.15}$$

with

$$\sum_\mu A_\mu^\lambda B_\nu^\mu = \delta_\nu^\lambda, \qquad \sum_\lambda A_\mu^\lambda B_\lambda^\kappa = \delta_\mu^\kappa. \tag{8.16}$$

Then one has, after differentiation,

$$\mathrm{d}\theta^\lambda = \sum_{\nu<\rho} \gamma_{\rho\nu}^\lambda \theta^\nu \wedge \theta^\rho, \qquad \gamma_{\rho\nu}^\lambda := \sum_{\kappa,\mu} \left(\frac{\partial A_\mu^\lambda}{\partial x^\kappa} - \frac{\partial A_\kappa^\lambda}{\partial x^\mu}\right) B_\nu^\kappa B_\rho^\mu, \tag{8.17}$$

where $\gamma_{\rho\nu}^\lambda$ are called the Hamel symbols [22]. Now introducing the variable π^λ by

$$\pi^\lambda = \sum_\mu A_\mu^\lambda \dot{x}^\mu,$$

and replacing local coordinates $(x, \dot{x})$ in TW by (x, π), we can express the Lagrangian $L(x, \dot{x})$ as

$$\widetilde{L}(x, \pi) = L(x, \dot{x}). \tag{8.18}$$

Then the Euler–Lagrange equations described in terms of $(x, \dot{x})$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}^{\lambda}} - \frac{\partial L}{\partial x^{\lambda}} = 0 \tag{8.19}$$

are put in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \widetilde{L}}{\partial \pi^{\nu}} + \sum_{\mu,\rho} \gamma^{\mu}_{\nu\rho} \frac{\partial \widetilde{L}}{\partial \pi^{\mu}} \pi^{\rho} - X_{\nu}(\widetilde{L}) = 0. \tag{8.20}$$

We wish to apply this type of equation to the Lagrangian for a multi-particle system. The $\mathrm{d}q^{\alpha}$, Θ^{a} and the $\left(\frac{\partial}{\partial q^{\alpha}}\right)^{*}$, K_a form local bases of 1-forms and of vector fields on $T\dot{M}$, respectively. Since the left-invariant 1-forms Ψ^{a} on $SO(3)$ satisfy the structure equations

$$\mathrm{d}\Psi^{c} = -\frac{1}{2}\sum_{a,b} \epsilon_{abc} \Psi^{a} \wedge \Psi^{b}, \tag{8.21}$$

the exterior derivative of Θ^{c} takes the form

$$\mathrm{d}\Theta^{c} = -\sum_{a<b} \epsilon_{abc} \Theta^{a} \wedge \Theta^{b} + \sum_{a,\alpha}\sum_{b} \epsilon_{abc} \Lambda^{b}_{\alpha} \Theta^{a} \wedge \mathrm{d}q^{\alpha} + \sum_{\alpha<\beta} \kappa^{c}_{\alpha\beta}\, \mathrm{d}q^{\alpha} \wedge \mathrm{d}q^{\beta}, \tag{8.22}$$

where

$$\kappa^{c}_{\alpha\beta} := \frac{\partial \Lambda^{c}_{\beta}}{\partial q^{\alpha}} - \frac{\partial \Lambda^{c}_{\alpha}}{\partial q^{\beta}} - \sum_{a,b} \epsilon_{abc} \Lambda^{a}_{\alpha} \Lambda^{b}_{\beta}, \tag{8.23}$$

which are components of the curvature tensor $K_{\alpha\beta}$ given in (3.13). The exterior derivative of $\mathrm{d}q^{\alpha}$ vanishes. Thus, one obtains the Hamel symbols in the form

$$\begin{aligned} &\gamma^{c}_{ba} = -\epsilon_{abc}, \qquad \gamma^{c}_{\alpha a} = \sum_{b} \epsilon_{abc} \Lambda^{b}_{\alpha}, \qquad \gamma^{c}_{\beta\alpha} = \kappa^{c}_{\alpha\beta}, \\ &\gamma^{\alpha}_{\lambda\mu} = 0 \qquad \text{for} \quad \lambda, \mu \in \{a, \alpha\}. \end{aligned} \tag{8.24}$$

We now write out the Euler–Lagrange equations (8.20) with the Hamel symbols (8.24) in terms of the local coordinates $q^{\alpha}, g, \dot{q}^{\alpha}, \pi^{a}$ of $T\dot{M}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \widetilde{L}}{\partial \dot{q}^{\alpha}} + \sum_{\alpha,\beta} \kappa^{a}_{\beta\alpha} \frac{\partial \widetilde{L}}{\partial \pi^{a}} \dot{q}^{\beta} + \sum_{a,b}\sum_{c} \epsilon_{abc} \Lambda^{c}_{\alpha} \frac{\partial \widetilde{L}}{\partial \pi^{a}} \pi^{b} - X_{\alpha}(\widetilde{L}) = 0, \tag{8.25a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \widetilde{L}}{\partial \pi^{a}} - \sum_{b,c} \epsilon_{abc} \frac{\partial \widetilde{L}}{\partial \pi^{b}} \pi^{c} - \sum_{b,\alpha}\sum_{c} \epsilon_{acb} \Lambda^{c}_{\alpha} \frac{\partial \widetilde{L}}{\partial \pi^{b}} \dot{q}^{\alpha} - X_{a}(\widetilde{L}) = 0. \tag{8.25b}$$

We note here that

$$K_a(\widetilde{L}) = \frac{\mathrm{d}}{\mathrm{d}t} \widetilde{L}(\cdot, g\mathrm{e}^{tR(e_a)}, \cdot)\bigg|_{t=0} = \frac{1}{2}\left\langle g^{-1}\frac{\partial \widetilde{L}}{\partial g} - \left(\frac{\partial \widetilde{L}}{\partial g}\right)^{T} g, R(e_a)\right\rangle, \tag{8.26}$$

so that, for $X_{\alpha} = \left(\frac{\partial}{\partial q^{\alpha}}\right)^{*}$ and for $X_a = K_a$, we have

$$X_{\alpha}(\widetilde{L}) = \frac{\partial \widetilde{L}}{\partial q^{\alpha}} - \frac{1}{2}\left\langle g^{-1}\frac{\partial \widetilde{L}}{\partial g} - \left(\frac{\partial \widetilde{L}}{\partial g}\right)^{T} g, \Lambda_{\alpha}\right\rangle, \tag{8.27a}$$

$$X_{a}(\widetilde{L}) = \frac{1}{2}\left\langle g^{-1}\frac{\partial \widetilde{L}}{\partial g} - \left(\frac{\partial \widetilde{L}}{\partial g}\right)^{T} g, R(e_a)\right\rangle, \tag{8.27b}$$

respectively. Hence, the Euler–Lagrange equations (8.25) turn out to be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\widetilde{L}}{\partial\dot{q}^\alpha}-\frac{\partial\widetilde{L}}{\partial q^\alpha}=\sum_\beta\frac{\partial\widetilde{L}}{\partial\boldsymbol{\pi}}\cdot\boldsymbol{\kappa}_{\alpha\beta}\dot{q}^\beta-\frac{\partial\widetilde{L}}{\partial\boldsymbol{\pi}}\cdot(\boldsymbol{\pi}\times\boldsymbol{\lambda}_\alpha)-\frac{1}{2}\left\langle g^{-1}\frac{\partial\widetilde{L}}{\partial g}-\left(\frac{\partial\widetilde{L}}{\partial g}\right)^T g,\,\Lambda_\alpha\right\rangle, \tag{8.28a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\widetilde{L}}{\partial\boldsymbol{\pi}}=\frac{\partial\widetilde{L}}{\partial\boldsymbol{\pi}}\times\boldsymbol{\pi}-\sum_\alpha\frac{\partial\widetilde{L}}{\partial\boldsymbol{\pi}}\times\boldsymbol{\lambda}_\alpha\dot{q}^\alpha+R^{-1}\left(g^{-1}\frac{\partial\widetilde{L}}{\partial g}-\left(\frac{\partial\widetilde{L}}{\partial g}\right)^T g\right), \tag{8.28b}$$

which are equivalent to equation (3.12). Moreover, if the Lagrangian is rotationally invariant, one has $g^{-1}\frac{\partial\widetilde{L}}{\partial g}-\left(\frac{\partial\widetilde{L}}{\partial g}\right)^T g=0$, so that the Euler–Lagrange equations (8.28) reduce to

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\widetilde{L}}{\partial\dot{q}^\alpha}-\frac{\partial\widetilde{L}}{\partial q^\alpha}=\sum_\beta\frac{\partial\widetilde{L}}{\partial\boldsymbol{\pi}}\cdot\boldsymbol{\kappa}_{\alpha\beta}\dot{q}^\beta-\frac{\partial\widetilde{L}}{\partial\boldsymbol{\pi}}\cdot(\boldsymbol{\pi}\times\boldsymbol{\lambda}_\alpha), \tag{8.29a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\widetilde{L}}{\partial\boldsymbol{\pi}}=\frac{\partial\widetilde{L}}{\partial\boldsymbol{\pi}}\times\boldsymbol{\pi}-\sum_\alpha\frac{\partial\widetilde{L}}{\partial\boldsymbol{\pi}}\times\boldsymbol{\lambda}_\alpha\dot{q}^\alpha, \tag{8.29b}$$

which are also equivalent to equation (3.20). For comparison, see also [4], in which Θ^a and K_a are replaced by $\omega^a=g\Theta^a g^{-1}$ and $J_a=\sum g_{ab}K_b$, respectively, for the description of the Euler–Lagrange equations.

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